Entropy of random quantum states

1990 J. Phys. A: Math. Gen. 23 L1247

(http://iopscience.iop.org/0305-4470/23/23/011)

J. Phys. A: Math. Gen. 23 (1990) L1247–L1251. Printed in the UK

LETTER TO THE EDITOR

Entropy of random quantum states

Kingsley R W Jones
School of Physics, University of Melbourne, Parkville 3052, Melbourne, Australia

Received 5 October 1990

Abstract. We use a simple generating function to calculate exactly the entropy of random quantum states for finite-dimensional Hilbert spaces over real, complex and quaternionic scalars. This allows us to extend our previous formula for the quantum correlation information of a state determination apparatus to include real and quaternionic von Neumann analysers.

Calculation of the entropy of random quantum states is a problem of some topical importance given the recent appearance of such quantities in aspects of the theory of quantum chaos [1–5] and in the newly elaborated theory of quantum inference [6, 7]. The required entropy is

$$H(\alpha,\beta) \equiv -d \int (|\langle\psi \mid \phi\rangle|^2)^\beta \log(|\langle\psi \mid \phi\rangle|^2)^\alpha \, \mathrm{d}\hat{\Omega}_{\tilde{\psi}} \tag{1}$$

where ψ and ϕ are d-dimensional state vectors in a finite-dimensional Hilbert space over one of the three classical associative division algebras $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{H}$ of dimension $\nu \equiv 1, 2$ or 4 respectively and $\alpha, \beta \in \boldsymbol{C}$ with $\mathrm{Re}\,\beta > -\nu/2$. The symbol $\mathrm{d}\hat{\Omega}_{\tilde{\psi}}$ denotes the unique, normalized, unitary invariant measure upon the pure state manifold of normalized state vectors ψ and we have included a leading scaling by the dimension d for numerical convenience.

There are now numerous published values for particular integrals of type (1) that have arisen in the literature of quantum chaos [1–5]. Methods are available to do an exact calculation [8, 9] but the current literature concentrates upon $H(1,1)$ and gives only the corresponding asymptotic results [1, 2], special cases [3] or makes unnecessary approximations [4, 5]. Aside from its useful application in quantum chaos the quantity (1) is needed to calculate the quantum correlation information [6] of a state determination scheme. Indeed, knowledge of the more general result $H(\beta,\beta)$ proves fundamental to calculations in the area of quantum inference so we feel it warrants a single exposition.

Here we shall calculate (1) quite directly and exactly using a very simple generating function. A subsidiary result is then an elementary extension to real and quaternionic Hilbert spaces of the equations for quantum inference. In this way we arrive at a new class of optimal state determination problems analogous to those discussed in [6, 7].

The letter is organized as follows. First we discuss possible realizations of the invariant measure $\mathrm{d}\hat{\Omega}_{\tilde{\psi}}$ and introduce a simple method of calculation for a restricted class of integrands. We then identify an appropriate generating function and calculate

0305-4470/90/231247+05$03.50

an exact expression for (1) and compare this with other results in the literature. Finally, we detail the connection with quantum inference.

The most transparent realization of $\mathrm{d}\hat{\Omega}_{\bar{\psi}}$ is offered by the following delta function prescription:

$$\int \bullet \, \mathrm{d}\hat{\Omega}_{\bar{\psi}} = \frac{1}{\mathcal{N}} \int \bullet \, \delta(1 - \bar{\psi}\psi) \, \mathrm{d}\psi \, \mathrm{d}\bar{\psi} \tag{2}$$

with $\mathrm{d}\psi \, \mathrm{d}\bar{\psi} \equiv \prod_{j=1}^{d} \prod_{k=1}^{\nu} \pi^{-1/2} \, \mathrm{d}x_{jk}$ and $\mathcal{N}$ the appropriate normalization. For each choice of $\boldsymbol{F}$ the real variables x_{jk} define components of the state vector ψ upon an orthonormal basis $|j\rangle$, $j \in [1, d]$ through the following family of relations:

$$\boldsymbol{R}: \quad \nu = 1 \qquad \langle \psi \mid j \rangle = x_{j1} \tag{3}$$

$$\boldsymbol{C}: \quad \nu = 2 \qquad \langle \psi \mid j \rangle = x_{j1} + \boldsymbol{i} x_{j2} \tag{4}$$

$$\boldsymbol{H}: \quad \nu = 4 \qquad \langle \psi \mid j \rangle = x_{j1} + \boldsymbol{i} x_{j2} + \boldsymbol{j} x_{j3} + \boldsymbol{k} x_{j4} \tag{5}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ obey the standard quaternion algebra.

That (2) supplies a unitary invariant measure is obvious for $\boldsymbol{R}$ and $\boldsymbol{C}$. That the suggestive generalization to $\boldsymbol{H}$ is indeed correct can be proved by showing that the Jacobian for a quaternionic unitary transformation is unity. That this is so is well known from early calculations of the volume of the classical groups and is independently verified in Adler's work upon quaternionic quantum field theory [11].

Here we adopt a simplified form of (2) that is appropriate for bilinear integrands $F(\psi, \bar{\psi})$ possesing a scaling parameter $\beta \in \boldsymbol{R}$ such that:

$$F(\lambda\psi, \overline{\lambda\psi}) = (\lambda\lambda^*)^{\beta} F(\psi, \bar{\psi}) \qquad \forall \lambda \in \boldsymbol{F} \tag{6}$$

where *star* denotes $\boldsymbol{F}$ conjugation and *bar* $\boldsymbol{F}$-transpose conjugation.

Demanding unitary invariance we find that

$$\int F(\psi, \bar{\psi}) \, \mathrm{d}\hat{\Omega}_{\bar{\psi}} = \frac{1}{\mathcal{N}(\beta, \nu, d)} \int F(\psi, \bar{\psi}) \mathrm{e}^{-\bar{\psi}\psi} \, \mathrm{d}\psi \, \mathrm{d}\bar{\psi}. \tag{7}$$

This result follows immediately on expressing the right-hand side in polar coordinates. Notice that $\mathcal{N}(\beta, \nu, d)$ is a purely geometric factor that depends only upon β and the choice of Hilbert space.

Choosing $F(\psi, \bar{\psi}) = (\bar{\psi}\psi)^{\beta}$, the left-hand side of (7) is unity, and we find that

$$\mathcal{N}(\beta, \nu, d) = \frac{\Gamma(\nu d/2 + \beta)}{\Gamma(\nu d/2)}. \tag{8}$$

This is readily verified for integral β and holds in general for $\mathrm{Re}\,\beta > -\nu d/2$ (only gamma functions are encountered). There is then a natural analytic continuation of the function $H(\alpha, \beta)$ in β and νd to $\nu d/2 + \beta \neq -1, -2, \ldots$ with a possibly non-integral *effective* dimension $f = \nu d$. However, the physical meaning of this is unclear.

Observe that

$$\int (|\langle \psi \mid \phi \rangle|^2)^{\beta} \log(|\langle \psi \mid \phi \rangle|^2)^{\alpha} \, \mathrm{d}\hat{\Omega}_{\bar{\psi}} = \alpha \frac{\mathrm{d}}{\mathrm{d}\beta} \int (|\langle \psi \mid \phi \rangle|^2)^{\beta} \, \mathrm{d}\hat{\Omega}_{\bar{\psi}}. \tag{9}$$

Then the required generating function is

$$H(\alpha,\beta) = -d\,\alpha \frac{\mathrm{d}}{\mathrm{d}\beta} \int (|\langle\psi \mid \phi\rangle|^2)^\beta \, \mathrm{d}\hat{\Omega}_{\tilde{\psi}}. \tag{10}$$

Making use of (7) and (8) it is then easy to show that:

$$S(\beta) \equiv d \int (|\langle\psi \mid \phi\rangle|^2)^\beta \, \mathrm{d}\hat{\Omega}_{\tilde{\psi}} = d\,\frac{\Gamma(\nu d/2)}{\Gamma(\nu/2)} \times \frac{\Gamma(\nu/2+\beta)}{\Gamma(\nu d/2+\beta)}. \tag{11}$$

Substituting this result into (10) we find

$$H(\alpha,\beta) = -d\,\alpha\,\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\frac{\Gamma(\nu d/2)}{\Gamma(\nu/2)} \times \frac{\Gamma(\nu/2+\beta)}{\Gamma(\nu d/2+\beta)}\right]. \tag{12}$$

Noting that $\Gamma'(z)/\Gamma(z) = \Psi(z)$, the digamma function, this expression readily reduces to the exact result

$$H(\alpha,\beta) = d\,\alpha\,\frac{\Gamma(\nu d/2)}{\Gamma(\nu/2)} \times \frac{\Gamma(\nu/2+\beta)}{\Gamma(\nu d/2+\beta)}\left\{\Psi(\nu d/2+\beta) - \Psi(\nu/2+\beta)\right\}. \tag{13}$$

Zyczkowski [4] and Wootters [5] have published approximate calculations of (1) based upon the Porter–Thomas result [10] of random matrix theory. This says that for d large and ψ uniformly distributed in Hilbert space, we have for the approximate distribution of components $y \equiv |\langle\psi \mid \phi\rangle|^2$, a χ^2 density:

$$P_\nu(y) = \left(\frac{\nu}{2\langle y\rangle}\right)^{\nu/2} \frac{1}{\Gamma(\nu/2)}\, y^{\nu/2-1} \exp(-\nu y/2\langle y\rangle) \tag{14}$$

where $\langle y\rangle = 1/d$.

Zyczkowski [4] used this to obtain:

$$H(1,1) \sim -d \int_0^\infty y \log y \, P_\nu(y)\,\mathrm{d}y = \log(d\nu/2) - \Psi(\nu/2+1). \tag{15}$$

Now, using $\Gamma(z+1) = z\Gamma(z)$, (13) reduces to an exact expression for Zyczkowski's integral (15):

$$H(1,1) = \Psi(\nu d/2+1) - \Psi(\nu/2+1). \tag{16}$$

Taking d large and using the asymptotic result: $\Psi(\nu d/2+1) \sim \log(\nu d/2) + 1/\nu d + \mathrm{O}(1/d^2)$, we find

$$H(1,1) \sim \log(\nu d/2) - \Psi(\nu/2+1) + 1/\nu d + \mathrm{O}(1/d^2).$$

So (15) indeed gives the correct asymptotics in the large-d limit. Furthermore, the error term $1/\nu d$ provides an interesting information theoretic measure of the convergence to asymptotic validity of the Porter–Thomas χ^2 approximation.

Aside from $H(1,1)$, Zyczkowski also calculates approximations to (13) and (11) for $H(\beta,\beta)$ and $S(\beta)$ to find:

$$S(e/2) \sim d^{(1-e)/2} \left(\frac{2}{\nu}\right)^{e/2} \frac{\Gamma(\nu/2+e/2)}{\Gamma(\nu/2)} \tag{17}$$

$$H(u,u) \sim d^{1-u} \left(\frac{2}{\nu}\right)^{u} \frac{\Gamma(\nu/2+u)}{\Gamma(\nu/2)} \{\log(\nu d/2) - \Psi(\nu/2+u)\}. \tag{18}$$

These compare quite well with our exact results upon making use of the large-d approximation

$$\Gamma(\nu d/2)/\Gamma(\nu d/2+\beta) \sim (\nu d/2)^{-\beta}.$$

Better asymptotics are possible, but we find it very interesting to note that the χ^2 approximation gets a number of terms in the exact formulae precisely.

An exact version of (14) is also available [9, 10]. Other authors [1, 2] have used this to obtain the correct asymptotics of $H(1,1)$. Casati *et al* [3] have previously given the true exact result for $H(1,1)$ at $\nu = 1$ (except $\Psi(1/2)$ should read $\Psi(3/2)$ in their paper). Also, in [7] we gave the exact result for $H(1,n)$ at $\nu = 2$ via a very different route to that pursued here. Although elementary, the full result (13) does not appear to have been published before.

In quantum chaos the entropy of a state appears as a measure of the dispersion generated in the evolution of the quantum analogues of classically chaotic systems. An entropy close to $H(1,1)$, with ν fixed by the appropriate universality class, is then a fair surmise for that of a typical *snapshot* state arising from the random behaviour of such a system in an ergodic-like region of phase space (particularly for large d).

In quantum inference $H(1,1)$ appears in a quite different way as an information bias we must remove in order to properly obtain the information gained about *a priori* unknown quantum states when we make succesive measurements upon an identical ensemble with a view to *measuring* the state. Wootters [5] has given a nice discussion of the subtleties of information theory in this regard.

In previous articles [6, 7] we gave a detailed exposition of the principles behind complex quantum inference. There is now no inherent difficulty in generalizing this work to real and quaternionic Hilbert space (in particular, the non-commutativity of $\boldsymbol{H}$ does not enter). Following [6], we simply allow real or quaternionic projectors as data, rather than complex ones. The quantum invariant prior becomes the appropriate version of (2) and we may transcribe directly from [6] the same inversion equations:

$$p(\psi \mid \Phi_N) = \frac{1}{p(\Phi_N)} \prod_{k=1}^{N} |\langle \psi \mid \phi_k \rangle|^2 \tag{19}$$

$$p(\Phi_N) = \int \prod_{k=1}^{N} |\langle \psi \mid \phi_k \rangle|^2 \, \mathrm{d}\hat{\Omega}_{\tilde{\psi}} \tag{20}$$

thereby giving an inferred distribution for the measured state parametrized by the N observed eigenprojectors $\Phi_N = \{|\phi_k\rangle\langle\phi_k|\}_{k=1}^{N}$ However, now $|\phi_k\rangle \in \boldsymbol{R}^d$, $\boldsymbol{C}^d$ or $\boldsymbol{H}^d$.

Again following [6] we associate to any real, complex or quaternionic von Neumann analyser the quantum correlation information

$$\{\psi, \Phi_N\} = -N\{\Psi(\nu d/2+1) - \Psi(\nu/2+1)\} - \sum_{\Phi_N} p(\Phi_N) \log p(\Phi_N).$$

Here, the only difference from the complex case is that we have replaced the quantity C_d of [6]) by its general cousin $H(1,1)$ evaluated for the $\boldsymbol{F}$ in question. Note further that the form given above makes plain the origin of Euler's constant, γ, in [6] since $\lim_{d\to\infty} \Psi(d+1) - \Psi(2) = \log d - (1-\gamma)$. Also we have the new crude bound:

$$\{\psi, \Phi_N\} \leq N\{\log d - H(1,1)\}$$

showing existence of three distinct classes of optimal state determination problems defined as the extremization of the entropy of $p(\Phi_N)$ with respect to the choice of N bases for $\boldsymbol{R}^d$, $\boldsymbol{C}^d$ or $\boldsymbol{H}^d$. Furthermore, these three classes exhaust (in a foundational sense) the available finite-dimensional Hilbert spaces [12] and are each known to occur (in a purely algebraic sense) for quantum systems with appropriate Hamiltonian symmetry [13]. Constraints upon knowledge of quantum states similar to those derived in [7] are indicated. These problems will be investigated thoroughly in a forthcoming paper [14].

I thank Bill Wootters for some very useful literature and for pointing out the relevance of some of my earlier results to current work upon informational aspects of quantum chaos. I also thank Andrew Davies for help with things quaternionic.

References

[1] Izrailev F M 1989 *J. Phys. A: Math. Gen.* **22** 865
[2] Scharf R 1989 *J. Phys. A: Math. Gen.* **22** 4223
[3] Casati G, Guarneri I, Izrailev F and Scharf R 1990 *Phys. Rev. Lett.* **64** 5
[4] Zyczkowski K 1990 *J. Phys. A: Math. Gen.* **23** 4427
[5] Wootters W K 1990 *Found. Phys.* in press
[6] Jones K R W 1990 *University of Melbourne Preprint* UM-P-90/31 OZ-90/11
[7] Jones K R W 1990 *University of Melbourne Preprint* UM-P-90/32 OZ-90/12; *J. Phys A: Math. Gen.* in press
[8] Porter C E and Rosenzweig N 1960 *Ann. Acad. Sci. Fennicae* **AVI 44** 5
[9] Kus M, Mostowski J and Haake F 1988 *J. Phys. A: Math. Gen.* **21** L1073
[10] Porter C E and Thomas R G 1956 *Phys. Rev.* **104** 483
[11] Adler S L 1986 *Quantum Field Theory and Quantum Statistics* vol 1, ed I A Batalin, C J Isham and G A Vilkovisky (Bristol: Hilger) p 601
[12] Jauch J M 1968 *Foundations of Quantum Mechanics* (New York: Addison-Wesley)
[13] Dyson F J 1962 *J. Math. Phys.* **3** 1199
[14] Davies A J and Jones K R W 1990 paper in preparation